

POINTLIKE HOPF DEFECTS IN ABELIAN PROJECTIONS

FALK BRUCKMANN

*Theoretisch-Physikalisches Institut, Friedrich-Schiller-Universität Jena
Max-Wien-Platz 1, D-07743 Jena
email: brk@tpi.uni-jena.de*

We present a new kind of defect in Abelian Projections, stemming from pointlike zeros of second order. The corresponding topological quantity is the Hopf invariant $\pi_3(S^2)$ (rather than the winding number $\pi_2(S^2)$ for magnetic monopoles). We give a visualisation of this quantity and discuss the simplest non-trivial example, the Hopf map. Such defects occur in the Laplacian Abelian gauge in a non-trivial instanton sector. For general Abelian projections we show how an ensemble of Hopf defects accounts for the instanton number.

1 Introduction

It has long been speculated that confinement in pure Yang-Mills theories may be realised via dual superconductivity¹. To arrive at this picture, 't Hooft has suggested to use Abelian projections². In this technique one fixes the gauge group up to its maximal Abelian subgroup. The generic defects of this gauge fixing are magnetic monopoles. They are supposed to play the role of dual Cooper pairs and force the chromoelectric field into strings.

Abelian projections are best described by an ‘auxiliary Higgs field’ ϕ in the adjoint representation. We will focus on the gauge group $SU(2)$ in the following. An Abelian gauge (AG) assigns to every field A a field ϕ in such a way that *the gauge transformation Ω which diagonalises ϕ is the one which brings A into the AG*. The unfixed $U(1)$ consists of rotations around the 3-direction in isospace. Defects of such a gauge fixing arise when ϕ vanishes at some \bar{x} : Ω is not well-defined there. For a topological description define $n \equiv \phi/|\phi|$ around \bar{x} , which then is a regular mapping onto $S^2 \cong SU(2)/U(1)$.

2 Zeros of the Higgs field

There are different types of zeros of ϕ . Generically, this field vanishes on lines/loops, since there are three equations to be solved on a four dimensional manifold. As an example consider $\vec{\phi} = \vec{x}$ with a first order zero. This ‘hedgehog’ field produces a static defect at the origin of \mathbb{R}^3 . The associated n is a mapping $S^2_{|\vec{x}| \text{ fixed}} \rightarrow S^2$, which is characterised by an integer winding number, $\pi_2(S^2) = \mathbb{Z}$. It counts how many times one sphere is covered by the

other and equals the magnetic charge of a monopole arising in the Abelian projected gauge field. When one performs the diagonalisation of ϕ , a Dirac string piercing any sphere around the defect is unavoidable.

Alternatively ϕ may vanish on isolated points. Consider $|\phi| = r^2$ which has a zero of second order at the origin. Now $n : S^3_{r^{\text{fixed}}} \rightarrow S^2$ gives rise to another topological quantity, the Hopf invariant, $\pi_3(S^2) = \mathbb{Z}$.

3 The Hopf invariant

The definition of the Hopf invariant via the homotopy group is rather abstract. A more intuitive form can be used under some regularity assumptions³. Then the preimage of a point on S^2 is a loop on S^3 . The Hopf invariant counts *how many times two such loops are linked*. This linking number can be understood via a combination of Biot-Savart's and Ampere's law⁴.

The topology of n and its diagonalising gauge transformation Ω are related. The latter is a mapping $S^3 \rightarrow S^3 \cong SU(2)$ with the usual winding number equal to the Hopf invariant. By encoding the topology in Ω it is possible to diagonalise n smoothly, there are no further Dirac strings.

4 The Hopf map

A nice example of a non-trivial mapping n is the Hopf map⁵. Take

$$\phi_a = \begin{pmatrix} 2(x_1x_3 + x_2x_4) \\ 2(x_2x_3 - x_1x_4) \\ x_1^2 + x_2^2 - x_3^2 - x_4^2 \end{pmatrix}, \quad |\phi| = r^2, \quad n = \phi/r^2 \equiv n_H. \quad (1)$$

It has a second order zero at the space-time origin and the corresponding n has Hopf invariant one. In order to visualise the latter, one views S^3 as compactified \mathbb{R}^3 (cf. Fig. 1). The preimages of the north and south pole are the z -axis and a circle in the xy -plane, respectively, being linked once. The energy density is proportional to the gradient of the spins, thus here it is localised near that circle. The configuration plays a role in a Skyrme-like model for low energy QCD, where it is called *torus-shaped unknot soliton*⁶.

This 'standard' mapping is diagonalised by another 'standard' mapping $\tilde{\Omega} = (x_4\mathbb{1} + ix_a\sigma_a)/r$, the identity $S^3 \rightarrow S^3$.

5 Significance for Abelian projections

We recently found Hopf defects in the Laplacian Abelian gauge (LAG)⁷. In this gauge the Higgs field ϕ is defined as the ground state of the covariant

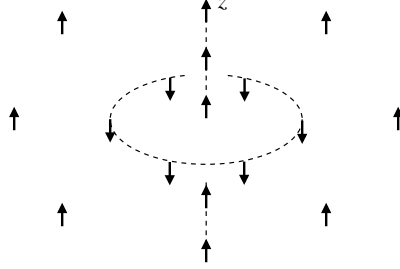


Figure 1. The Hopf map visualised as a system of spins with normalised length (S^2) in compactified space ($\mathbb{R}^3 \simeq S^3$).

Laplacian $-D^2[A]$ in the background of A . In order to avoid a pure scattering spectrum, one better works in a finite volume space-time say the sphere or the torus. On S^4 the fibre bundle construction of the 't Hooft instanton consists of two patches around the north and south pole with A in singular and regular gauge, respectively (cf. Fig. 2). Inbetween they are related by the gauge transformation $\tilde{\Omega}$. Demanding ϕ to have the same transition function $\tilde{\Omega}$, n comes out to be σ_3 and n_H over these patches, which gives a Hopf defect at the instanton core (the origin). In order to diagonalise n one has to apply $\tilde{\Omega}$, which also transforms A^{reg} into A^{sg} . The result is the A field in singular gauge everywhere^a.

The Hopf defect may also be seen as a (twisted) *monopole loop with vanishing radius*. Thus a generic perturbation of the instanton induces a monopole loop again⁸. In the same manner one can understand the occurrence of two monopole loops for the instanton-anti-instanton in the LAG⁹.

For general Abelian gauges the discussion of defects is based on the Hopf invariant¹⁰: Any localised defect (points, loops or else) can be enclosed by a sphere. There n and its Hopf invariant can be computed. Like in residue calculus, a sphere containing no defect cannot carry a Hopf invariant. Thus the signed sum of all Hopf invariants gives the Hopf invariant at the boundary/in the transition region, which is exactly the instanton number,

$$\sum_i \text{Hopf}|_{S_i^3}(n) = \text{instanton number}(A). \quad (2)$$

That is, in a non-trivial background there *must* be defects. However, some of the defects may cancel in the sum. For the instanton in the LAG only the

^alike in the maximal Abelian gauge

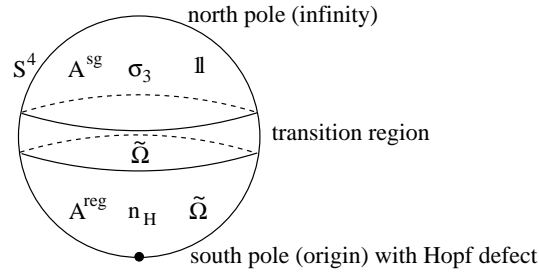


Figure 2. The one-instanton on the sphere, its Higgs field and diagonalising gauge transformation in the LAG.

minimal number of defects arises.

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